SISTEMA INTERAMERICANO DE METROLOGÍA

SIM METROLOGY SCHOOL

Uncertainty - MWG13/CCU

SPEAKER

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Metrology Working Group 13 – Statistics and Uncertainty

Why is important this group?

Because provides support to the other SIM Working Groups in all matters that relate to statistical modeling, statistical data analysis, and uncertainty evaluation, including for proficiency testing, interlaboratory studies and key comparisons.

Who is the chair?

Antonio Possolo (antonio.possolo@nist.gov)

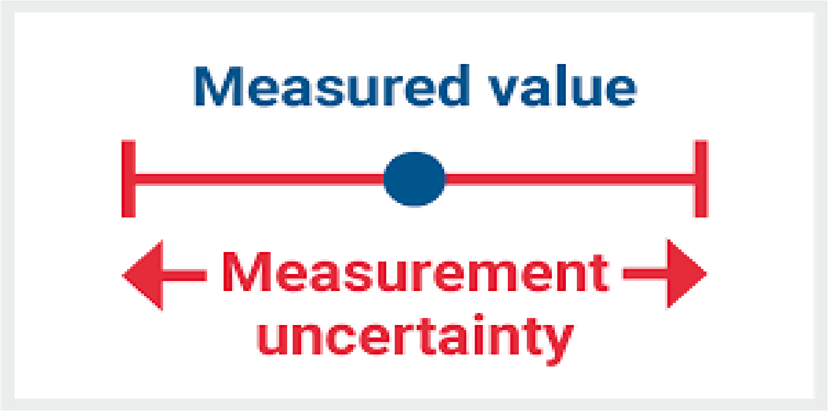
Measurement Uncertainty

What is Mesurement?

A measured value is an estimate of the true value of a property, which may be quantitative or qualitative.

What is Mesurement uncertainty?

Measurement uncertainty is the doubt about the true value of the measurand that remains after making a measurement

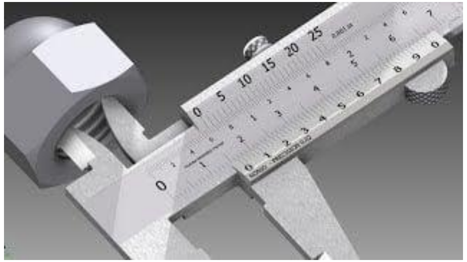


Probability

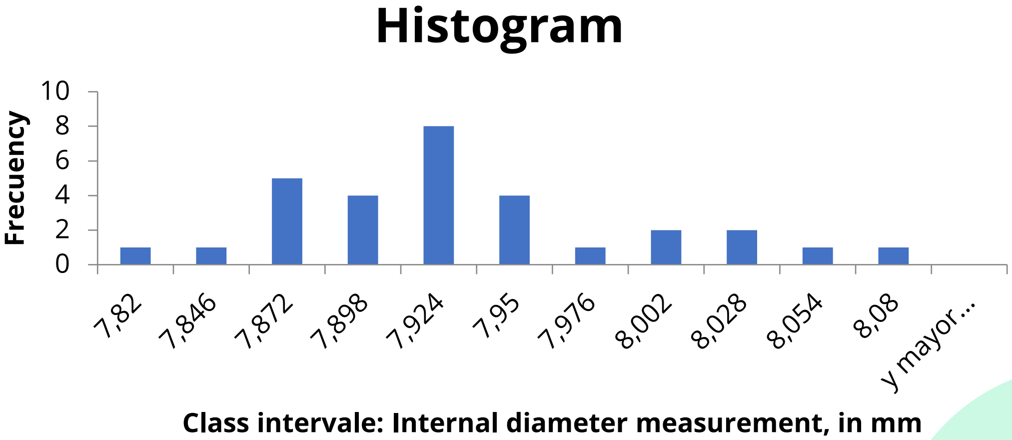
What is probability?

In a simple way it is the measure of the belief that a future event can occur.

- **Experiment**: Flip a coin
- **Event**: See obverse of the coin
- **Probability**: 0,9 (for an unbalanced coin or not fair coin)



Clase	Frecuencia
7,820	1
7,846	1
7,872	5
7,898	4
7,924	8
7,950	4
7,976	1
8,002	2
8,028	2
8,054	1
8,080	1



Probability distributions

Probability distributions

In Statistics, the probability distribution gives the possibility of each outcome of a random experiment or event.

Discrete probability distribution

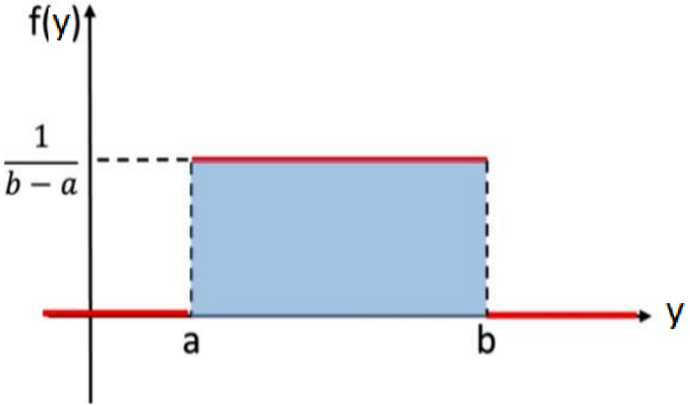
- Binomial probability distribution
- Geometric probability distribution
- Hipergeometric probability distribution
- Poisson probability distribution

Continuous probability distribution

- Uniform probability distribution
- Normal probability distribution
- Triangular probability distribution
- T probability distribution

Probability distributions

Uniform probability distribution



$$\mathrm{E}[X] = \int_a^b \frac{x}{b-a}\,dx = \frac{a+b}{2}$$

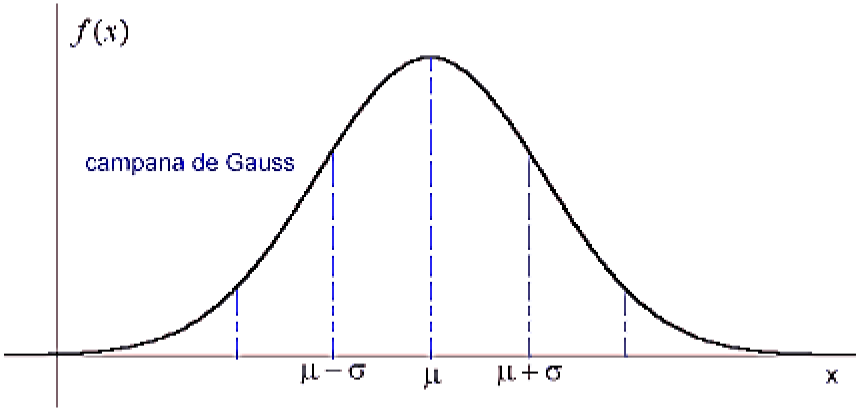
$$\mathrm{Var}(X) = E(X^2) - E(X)^2$$

$$f(y) = \begin{cases} \dfrac{1}{b-a}, a \leq y \leq b \\ 0, en\ cualquier\ otro\ valor \end{cases}$$

$$\mu = E(Y) = \frac{b+a}{2}$$

$$\sigma^2 = V(Y) = \frac{(b-a)^2}{12}$$

Probability distributions
Normal probability distribution



$$f(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(y-\mu)^2/(2\sigma^2)}, \qquad -\infty < y < \infty$$

$$E(Y) = \mu$$

$$V(Y) = \sigma^2$$

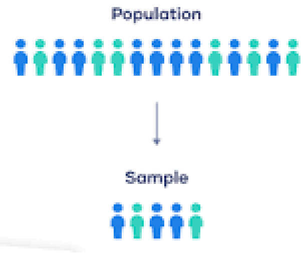
Estimation theory

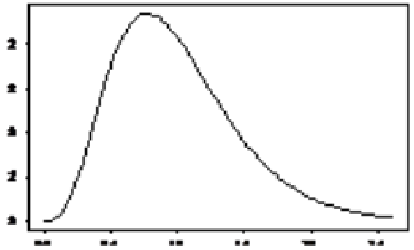
Inferential statistics

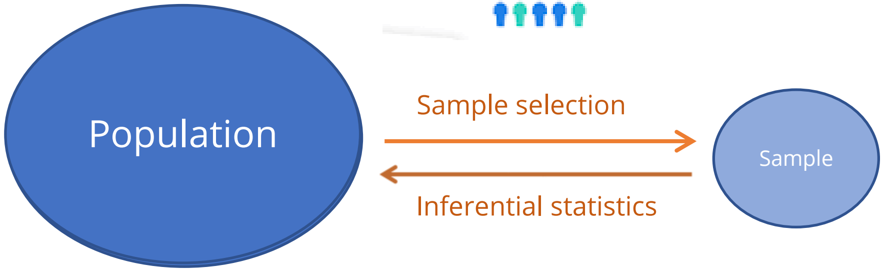
Inferential statistics

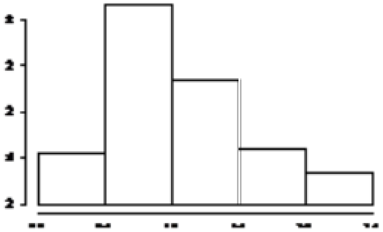
What does inferential statistic look for?

Help you come to conclusions and make predictions based on your data.









Inferential statistics

Parameters estimation

When we have unknown parameters of one or several populations, we can estimate it using observed data.

The estimate can be:

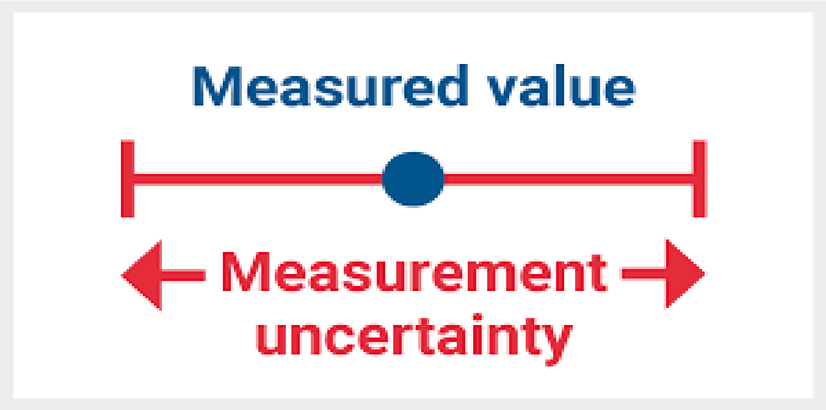
a) **Point estimate:** This estimation is a single value estimate of a parameter. (So, it's a number)

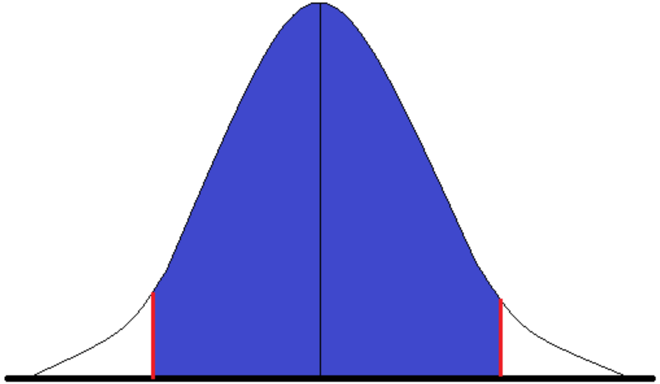
Estimator	Estimate to	Parameter
$\bar{X}$	Estimate to	μ
S^2	Estimate to	σ^2
P	Estimate to	π
$\bar{X}_1 - \bar{X}_2$	Estimate to	$\mu_1 - \mu_2$
$P_1 - P_2$	Estimate to	$\pi_1 - \pi_2$
$\bar{X}_d$	Estimate to	μ_d

Inferential statistics

Parameters estimation

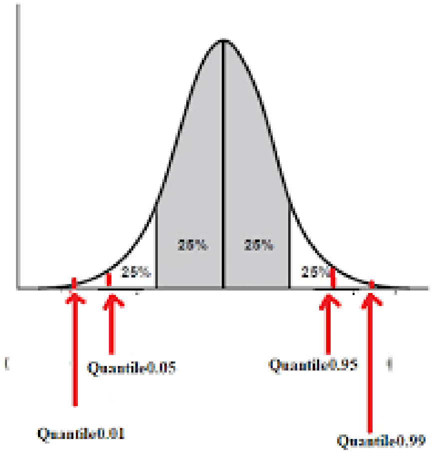
b) **Interval estimate:** It gives you a range of values where the parameter is expected to lie.





Inferential statistics
Parameters estimation

Estimation of intervals from a normal distribution



- For a population mean (μ)

When σ^2 is known:

$$IC(\mu) = \left[\bar{X} \pm Z_{(1-\alpha/2)}\sigma_{\bar{X}}\right]$$

$Z_{(1-\alpha/2)}$ It is the quantile of the Standard Normal Distribution with probability level $(1-\alpha/2)$. **In R Studio software, the function that you can used is z.test of the package BSDA**.

When σ^2 is unknown:

$$IC(\mu) = \left[\bar{X} \pm t_{(1-\alpha/2,n-1)}S_{\bar{X}}\right]$$

$t_{(1-\alpha/2,n-1)}$ It is the quantile of the T-distribution with probability level $(1-\alpha/2)$ and n-1 degrees of freedom. **En R se puede usar la función t.test.**

Measurand

Measurand

Quantity intended to be measured

NOTE

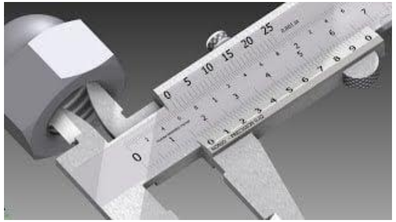
The measurement might change the phenomenon, body, or substance such that the quantity being measured may differ from the measurand. In this case adequate correction is necessary.

Examples:

- Vapor pressure of a given sample of water at 20 °C .
- The length of a steel rod in equilibrium with the ambient temperature of 23 °C .

Measurement Model

$$h(Y, X_1, X_{2,\ldots,} X_n) = 0$$



Measurement model

Mathematical relation among all quantities known to be involved in a measurement.

$$Y = f(X_1, X_{2,\ldots,} X_n)$$

Input quantities

Process of experimentally obtaining one or more quantity values that can reasonably be attributed to a quantity

$$L = \bar{I}_v + \delta_{cert} + \delta_{res} + \delta_{\Delta T}$$

Measurement error and uncertainty

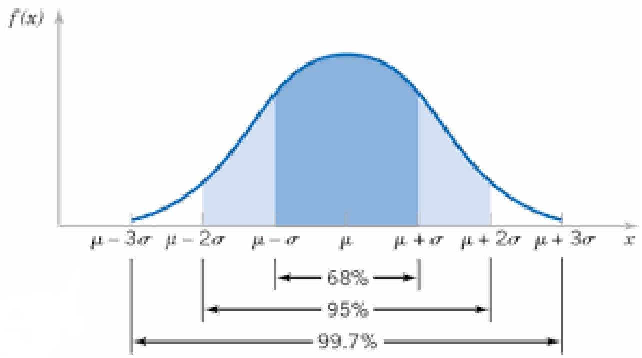
Measurement error

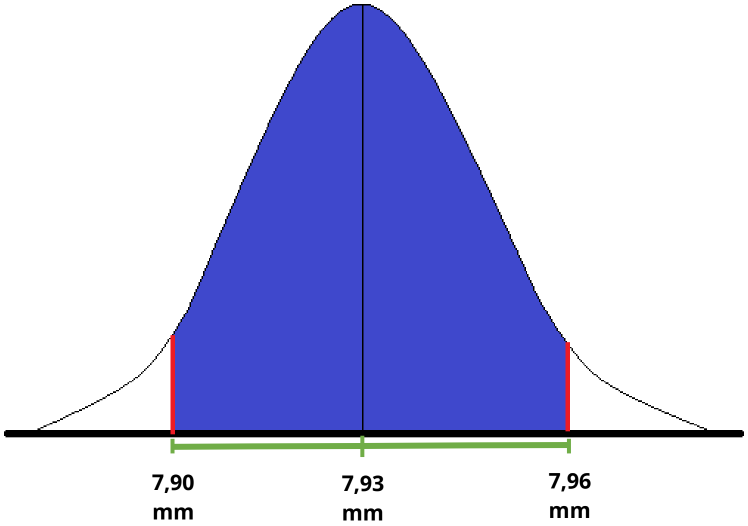
Measured quantity value minus a reference quantity value.

$$\textit{Measurement error} = \textit{Equipment indication} - \textit{Indication or value of the measurement standard}$$

Measurement uncertainty

Non-negative parameter characterizing the dispersion of the quantity values being attributed to a measurand, based on the information used.





Measurement uncertainty

Type A evaluation of measurement uncertainty

Evaluation of a component of measurement uncertainty by a statistical analysis of measured quantity values obtained under defined measurement conditions.

Type B evaluation of measurement uncertainty

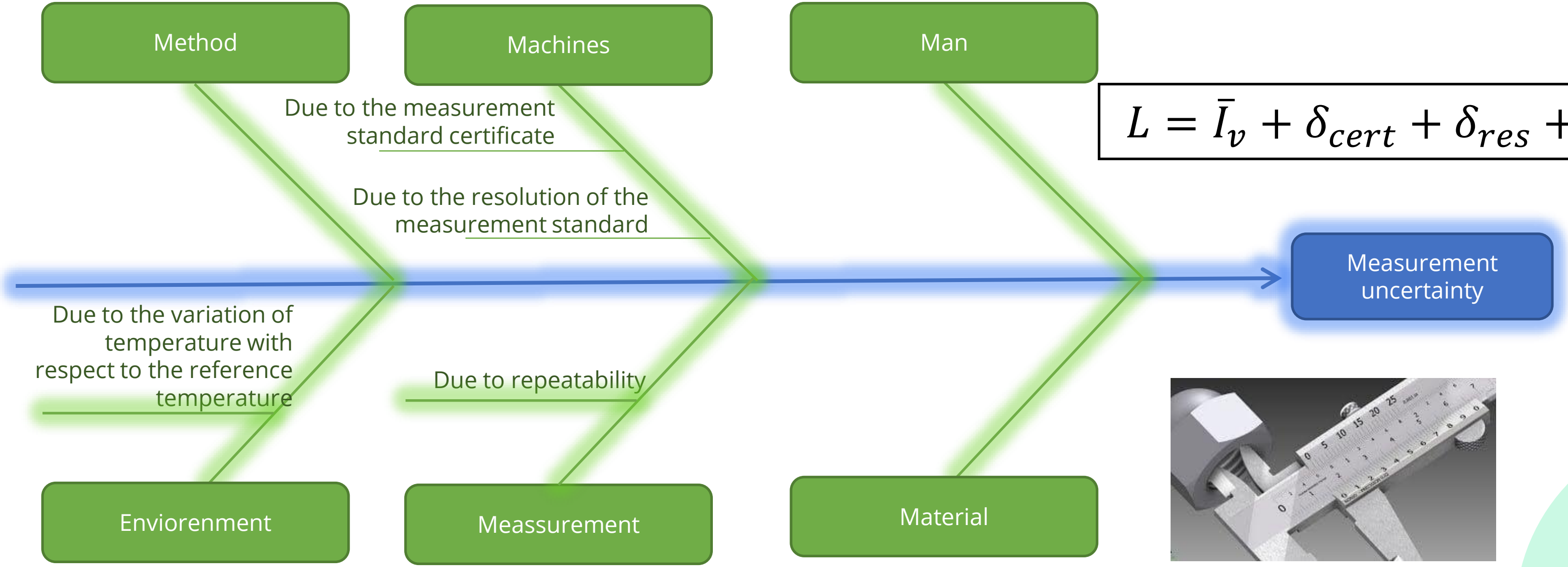
Evaluation of a component of measurement uncertainty determined by means other than a Type A evaluation of measurement uncertainty.

- associated with authoritative published quantity values;
- associated with the quantity value of a certified reference material;
- obtained from a calibration certificate;
- about drift;
- obtained from the accuracy class of a verified measuring instrument;
- obtained from limits deduced through personal experience.

Tools for determination sources of uncertainty

Diagram of Ishikagua

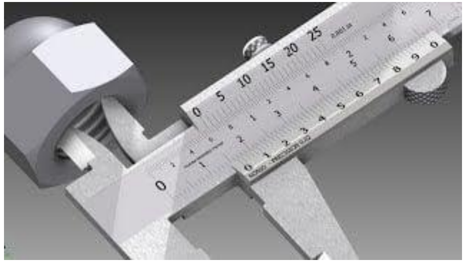
It is a visual tool that has a graphic format. It is typically used to identify possible causes of a specific problem.



$$L = \bar{I}_v + \delta_{cert} + \delta_{res} + \delta_{\Delta T}$$

Estimation of measurement uncertainty

Coefficient of expansion α (1/°C)	1,20E-05
Reference temperature(°C)	20
Measurement temperature(°C)	23
Vernier Resolution(mm)	0,01
Correction by certificate(mm)	0,01
Ucert (mm)	0,01



Standard measurement uncertainty

Measurement uncertainty expressed as a standard deviation

Measurement model: $$L = \bar{I}_v + \delta_{cert} + \delta_{res} + \delta_{\Delta T}$$

Imput quantities	Estimation or expected value	Estimation or espected value (mm)	Standard measurement uncertainty"u" (mm)
Repeatability	$\bar{I}_v$	7,92	$\frac{1}{\sqrt{n}}\sqrt{\frac{(I_i - \bar{I}_v)^2}{n-1}} = 0{,}011$
Due to the calibration certificate of the measuring standard	δ_{cert}	0,01	$\frac{U}{2} = \frac{0{,}013\ mm}{2} = 0{,}006\ 5$
Due to the resolution of the measurement standard	δ_{res}	0	$\frac{res}{2\sqrt{3}} = 0{,}003$
Due to temperature variation	$\delta_{\Delta T}$	0	$\frac{\alpha \Delta T \bar{I}_v}{\sqrt{3}} = 1{,}65x10^{-4}$

Sensitivity coefficients

$$L = \bar{I}_v + \delta_{cert} + \delta_{res} + \delta_{\Delta T}$$

Describe how the output estimate *y* varies with changes in the values of the input estimates.
If this change is generated by the standard uncertainty of the estimate xi , the corresponding variation in *y* is:

$$(\partial f / \partial x_i) u(x_i)$$

$$u_c^2(y) = \sum_{i=1}^{N} [c_i u(x_i)]^2 \equiv \sum_{i=1}^{N} u_i^2(y)$$

Imput quantities	Estimation or espected value	Estimation or espected value (mm)	Sensivity coefficients	Standard measurement uncertainty"u" (mm)
Repeatability	$\bar{I}_v$	7,92	$\frac{\partial L}{\partial \bar{I}_v} = 1$	$urep = \frac{1}{\sqrt{n}} \sqrt{\frac{(I_i - \bar{I}_v)^2}{n-1}} = 0{,}011$
Due to the calibration certificate of the measuring standard	δ_{cert}	0,01	$\frac{\partial L}{\partial \delta_{cert}} = 1$	$ucert = \frac{U}{2} = \frac{0{,}013\ mm}{2} = 0{,}006\ 5$
Due to the resolution of the measurement standard	δ_{res}	0	$\frac{\partial L}{\partial \delta_{res}} = 1$	$ures = \frac{res}{2\sqrt{3}} = 0{,}003$
Due to temperature variation	$\delta_{\Delta T}$	0	$\frac{\partial L}{\partial \delta_{\Delta T}} = 1$	$u\Delta T = \frac{\alpha \Delta T \bar{I}_v}{\sqrt{3}} = 1{,}65x10^{-4}$

Combined standard measurement uncertainty

Standard measurement uncertainty that is obtained using the individual standard measurement uncertainties associated with the input quantities in a measurement model.

Imput quantities	Estimation or espected value	Estimation or espected value (mm)	Sensivity coefficients	Standard measurement uncertainty"u" (mm)
Repeatability	$\bar{I}_v$	7,92	$\frac{\partial L}{\partial \bar{I}_v} = 1$	$urep = \frac{1}{\sqrt{n}}\sqrt{\frac{(I_i - \bar{I}_v)^2}{n-1}} = 0{,}011$
Due to the calibration certificate of the measuring standard	δ_{cert}	0,01	$\frac{\partial L}{\partial \delta_{cert}} = 1$	$ucert = \frac{U}{2} = \frac{0{,}013\ mm}{2} = 0{,}006\ 5$
Due to the resolution of the measurement standard	δ_{res}	0	$\frac{\partial L}{\partial \delta_{res}} = 1$	$ures = \frac{res}{2\sqrt{3}} = 0{,}003$
Due to temperature variation	$\delta_{\Delta T}$	0	$\frac{\partial L}{\partial \delta_{\Delta T}} = 1$	$u\Delta T = \frac{\alpha \Delta T \bar{I}_v}{\sqrt{3}} = 1{,}65x10^{-4}$

$$ucomb = \sqrt{urep^2 + ucert^2 + ures^2 + u\Delta T^2}$$

Expanded standard measurement uncertainty

Expanded uncertainty product of a combined standard measurement uncertainty and a factor larger than the number one.

Imput quantities	Estimation or espected value	Estimation or espected value (mm)	Sensivity coefficients	Standard measurement uncertainty"u" (mm)
Repeatability	$\bar{I}_v$	7,92	$\frac{\partial L}{\partial \bar{I}_v} = 1$	$urep = \frac{1}{\sqrt{n}}\sqrt{\frac{(I_i - \bar{I}_v)^2}{n-1}} = 0{,}011$
Due to the calibration certificate of the measuring standard	δ_{cert}	0,01	$\frac{\partial L}{\partial \delta_{cert}} = 1$	$ucert = \frac{U}{2} = \frac{0{,}013\ mm}{2} = 0{,}006\ 5$
Due to the resolution of the measurement standard	δ_{res}	0	$\frac{\partial L}{\partial \delta_{res}} = 1$	$ures = \frac{res}{2\sqrt{3}} = 0{,}003$
Due to temperature variation	$\delta_{\Delta T}$	0	$\frac{\partial L}{\partial \delta_{\Delta T}} = 1$	$u\Delta T = \frac{\alpha \Delta T \bar{I}_v}{\sqrt{3}} = 1{,}65x10^{-4}$

$$\boldsymbol{ucomb = \sqrt{urep^2 + ucert^2 + ures^2 + u\Delta T^2} = 0.013\ mm}$$

$$\boldsymbol{Uexp = k \times ucomb}$$

$\boldsymbol{k = 2\ for\ a}$ 95% of confidence interval

Measurement Uncertainty Budget

It is an itemized table of components that contribute to the uncertainty in measurement results.
Which has the greatest contribution of uncertainty?

Imput quantities	Estimation or espected value	Sensivity coefficients	Standard measurement uncertainty"u" (mm)	Contribution in %
Repeatability	$\bar{I}_v$	$\frac{\partial L}{\partial \bar{I}_v} = 1$	$urep = \frac{1}{\sqrt{n}}\sqrt{\frac{(I_i - \bar{I}_v)^2}{n-1}} = 0{,}011$	$\frac{urep^2}{ucomb^2} \times 100 = 71.6$
Due to the calibration certificate of the measuring standard	δ_{cert}	$\frac{\partial L}{\partial \delta_{cert}} = 1$	$ucert = \frac{U}{2} = \frac{0{,}013\ mm}{2} = 0{,}006\ 5$	$\frac{ucert^2}{ucomb^2} \times 100 = 25.0$
Due to the resolution of the measurement standard	δ_{res}	$\frac{\partial L}{\partial \delta_{res}} = 1$	$ures = \frac{res}{2\sqrt{3}} = 0{,}003$	$\frac{ures^2}{ucomb^2} \times 100 = 5.3$
Due to temperature variation	$\delta_{\Delta T}$	$\frac{\partial L}{\partial \delta_{\Delta T}} = 1$	$u\Delta T = \frac{\alpha \Delta T \bar{I}_v}{\sqrt{3}} = 1{,}65x10^{-4}$	$\frac{u\Delta T^2}{ucomb^2} \times 100 = 0.0$

$$ucomb = \sqrt{urep^2 + ucert^2 + ures^2 + u\Delta T^2} = 0.013\ mm$$

$$Uexp = k \times ucomb$$

$k = 2$ $for\ a$ 95% of confidence interval

Example level 2

The measurement model for the resistance of an element of an electrical circuit is: $R = (V/I)\cos(\phi)$

The estimates and standard uncertainties of the input quantities are:

QUANTITY	x	$u(x)$
V	4.9990 V	0.0032 V
I	19.6610×10^{-3} A	0.0095×10^{-3} A
ϕ	1.04446 rad	0.00075 rad

$$R = \left(\frac{4.9990\ V}{19.6610E-3}\right) \times \cos(1.04446) = 127.73\ \Omega$$

Imput quantities	Estimation or espected value	Estimation or espected value	Sensivity coefficients	Standard measurement uncertainty"u" (mm)
Voltage	V	4.9990 V	$\frac{\partial R}{\partial V} = \frac{\cos(\emptyset)}{I} = 25.55\ A^{-1}$	$u_V = 0.0032\ V$
Current	I	19.6610E-3 A	$\frac{\partial R}{\partial I} = -\left(\frac{V}{I^2}\right) \times \cos(\emptyset) = -6496.73\ V/A^2$	$u_I = 0.0095E-3\ A$
Angle	$\emptyset$	1.044 46 rad	$\frac{\partial R}{\partial \emptyset} = -\frac{V}{I} sen(\emptyset) = -219.85\ V/A$	$u_\emptyset = 0.00075\ rad$

Example level 2

Imput quantities	Estimation or espected value	Sensivity coefficients	Standard measurement uncertainty"u" (mm)	Contribution in %
Voltage	V	$\frac{\partial R}{\partial V} = \frac{\cos(\emptyset)}{I} = 25.55\ A^{-1}$	$u_V = 0.0032\ V$	$\frac{\left(\frac{\partial R}{\partial V}\right)^2 \times urep^2}{ucomb^2} \times 100 = 17.7$
Current	I	$\frac{\partial R}{\partial I} = -\left(\frac{V}{I^2}\right) \times \cos(\emptyset) = -6496.73\ V/A^2$	$u_I = 0.0095E-3\ A$	$\frac{\left(\frac{\partial R}{\partial I}\right)^2 \times ucert^2}{ucomb^2} \times 100 = 10.0$
Angle	$\emptyset$	$\frac{\partial R}{\partial \emptyset} = -\frac{V}{I} sen(\emptyset) = -219.85\ V/A$	$u_\emptyset = 0.00075\ rad$	$\frac{\left(\frac{\partial R}{\partial \emptyset}\right)^2 \times ures^2}{ucomb^2} \times 100 = 72.1$

$$ucomb = \sqrt{\left(\frac{\partial R}{\partial V}\right)^2 u_V{}^2 + \left(\frac{\partial R}{\partial I}\right)^2 u_I{}^2 + \left(\frac{\partial R}{\partial \emptyset}\right)^2 u_\emptyset{}^2} = 0.19\ \Omega$$

$$Uexp = k \times ucomb$$

$k = 2$ *for a* 95% of confidence interval

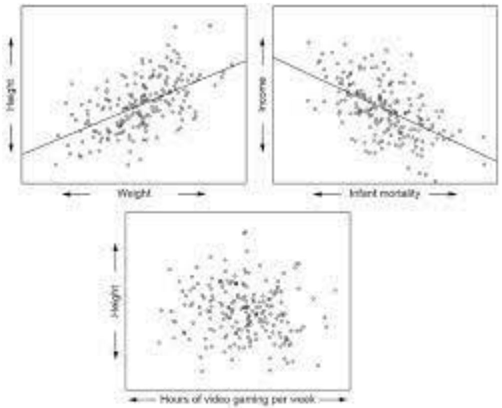
What happens if there is correlation between variables?

When the input quantities are correlated, the appropriate expression for the combined variance uc^2(y) associated with the measurement result is:

$$u_c^2(y)=\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}u(x_i,x_j)=\sum_{i=1}^{N}\left[\frac{\partial f}{\partial x_i}\right]^2u^2(x_i)+2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}u(x_i,x_j)$$

$$u_c^2(y)=\sum_{i=1}^{N}c_i^2u^2(x_i)+2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}c_ic_ju(x_i)u(x_j)r(x_i,x_j)$$

A correlation is a statistical measure of the relationship between two variables.



Example level 2

The measurement model for the resistance of an element of an electrical circuit is: $R = (V/I)\cos(\phi)$
The estimates and standard uncertainties of the input quantities and the correlations between them, are listed in the following table:

QUANTITY	x	$u(x)$
V	4.9990 V	0.0032 V
I	19.6610×10^{-3} A	0.0095×10^{-3} A
ϕ	1.04446 rad	0.00075 rad
$r(V,I) = -0.36$	$r(V,\phi) = 0.86$	$r(I,\phi) = -0.65$

$$R = \left(\frac{4.9990\ V}{19.6610E-3}\right) \times cos(1.04446) = 127.73\ \Omega$$

$$c(V,I) = 2 \times \frac{\partial R}{\partial V} \times \frac{\partial R}{\partial I} \times u_V \times u_I \times r(V,I) = 0.0036\ \Omega$$

$$c(V,\emptyset) = 2 \times \frac{\partial R}{\partial V} \times \frac{\partial R}{\partial \emptyset} \times u_V \times u_\emptyset \times r(V,\emptyset) = -0.0232\ \Omega$$

$$c(I,\emptyset) = 2 \times \frac{\partial R}{\partial I} \times \frac{\partial R}{\partial \emptyset} \times u_I \times u_\emptyset \times r(I,\emptyset) = -0.0132\ \Omega$$

Imput quantities	Estimation or espected value	Estimation or espected value	Sensivity coefficients	Standard measurement uncertainty"u" (mm)
Voltage	V	4.9990 V	$\frac{\partial R}{\partial V} = \frac{\cos(\emptyset)}{I} = 25.55\ A^{-1}$	$u_V = 0.0032\ V$
Current	I	19.6610E-3 A	$\frac{\partial R}{\partial I} = -\left(\frac{V}{I^2}\right) \times \cos(\emptyset) = -6496.73\ V/A^2$	$u_I = 0.0095E-3\ A$
Angle	$\emptyset$	1.044 46 rad	$\frac{\partial R}{\partial \emptyset} = -\frac{V}{I} sen(\emptyset) = -219.85\ V/A$	$u_\emptyset = 0.00075\ rad$

Example level 2

Imput quantities	Estimation or espected value	Sensivity coefficients	Standard measurement uncertainty"u" (mm)	Contribution in %
Voltage	V	$\frac{\partial R}{\partial V} = \frac{\cos(\emptyset)}{I} = 25.55\ A^{-1}$	$u_V = 0.0032\ V$	$\frac{\left(\frac{\partial R}{\partial V}\right)^2 \times \boldsymbol{urep}^2}{\boldsymbol{ucomb}^2} \times 100 = 136.5$
Current	I	$\frac{\partial R}{\partial I} = -\left(\frac{V}{I^2}\right) \times \cos(\emptyset) = -6496.73\ V/A^2$	$u_I = 0.0095E - 3\ A$	$\frac{\left(\frac{\partial R}{\partial I}\right)^2 \times \boldsymbol{ucert}^2}{\boldsymbol{ucomb}^2} \times 100 = 77.8$
Angle	$\emptyset$	$\frac{\partial R}{\partial \emptyset} = -\frac{V}{I} sen(\emptyset) = -219.85\ V/A$	$u_\emptyset = 0.00075\ rad$	$\frac{\left(\frac{\partial R}{\partial \emptyset}\right)^2 \times \boldsymbol{ures}^2}{\boldsymbol{ucomb}^2} \times 100 = 555.2$
Correlation	0	–	$\boldsymbol{2c}(\boldsymbol{V}, \boldsymbol{I}) + \boldsymbol{2c}(\boldsymbol{V}, \emptyset) + \boldsymbol{2c}(\boldsymbol{I}, \emptyset) = -0.033$	$\frac{\boldsymbol{2c}(\boldsymbol{V}, \boldsymbol{I}) + \boldsymbol{2c}(\boldsymbol{V}, \emptyset) + \boldsymbol{2c}(\boldsymbol{I}, \emptyset)}{ucomb^2} = -669.5$

$$\boldsymbol{ucomb} = \sqrt{\left(\frac{\partial R}{\partial V}\right)^2 u_V{}^2 + \left(\frac{\partial R}{\partial I}\right)^2 u_I{}^2 + \left(\frac{\partial R}{\partial \emptyset}\right)^2 u_\emptyset{}^2 + \boldsymbol{2c}(\boldsymbol{V}, \boldsymbol{I}) + \boldsymbol{2c}(\boldsymbol{V}, \emptyset) + \boldsymbol{2c}(\boldsymbol{I}, \emptyset)} = \boldsymbol{0.07\ \Omega}$$

$$\boldsymbol{Uexp} = \boldsymbol{k} \times \boldsymbol{ucomb}$$

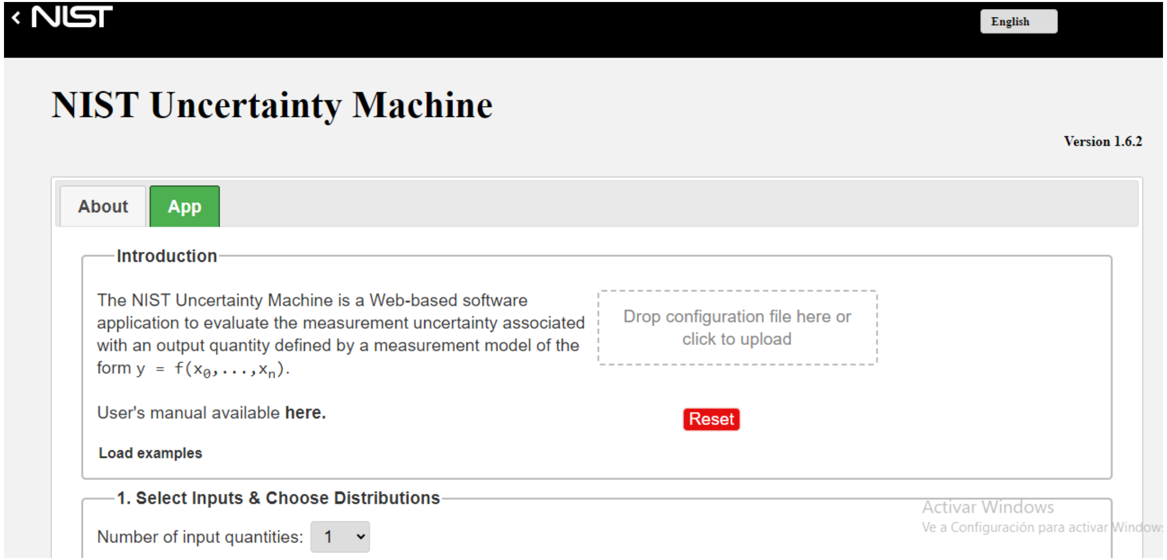
$\boldsymbol{k} = \boldsymbol{2}\ \boldsymbol{for\ a}$ 95% of confidence interval

Have you ever used the Uncertainty Machine from NIST?

Uncertainty Machine from NIST

https://uncertainty.nist.gov/

We can use this app to applicate in our examples.



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Thanks!

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